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# Two-sided vector spaces

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## Abstract

We study the structure of two-sided vector spaces over a perfect field  $K$ . In particular, we give a complete characterization of isomorphism classes of simple two-sided vector spaces which are left finite-dimensional. Using this description, we compute the Quillen  $K$ -theory of the category of left finite-dimensional, two-sided vector spaces over  $K$ . We also consider the closely related problem of describing homomorphisms  $\phi : K \rightarrow M_n(K)$ .

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## 1. Introduction

Given the central role that vector spaces play in mathematics, it is natural to study two-sided vector spaces; that is, abelian groups  $V$  equipped with both a left and right action by a field  $K$ , subject to the associativity condition  $(xv)y = x(vy)$  for  $x, y \in K$  and  $v \in V$ . When the left and right actions of  $K$  on  $V$  agree, then  $V$  is nothing more than an ordinary  $K$ -vector space. In this case,  $V$  decomposes into a direct sum of irreducible subspaces, and every irreducible subspace is 1-dimensional (and hence isomorphic to  $K$  as a vector space over  $K$ ). When the left and right

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actions of  $K$  and  $V$  differ, then the structure of  $V$  can be much more complicated. For example,  $V$  does not generally decompose into irreducible subspaces. Furthermore, the distinct irreducible subspaces of  $V$  may not be 1-dimensional or isomorphic to each other.

Apart from being intrinsically interesting, two-sided vector spaces play an important role in noncommutative algebraic geometry. In particular, two-sided vector spaces are noncommutative analogues of vector bundles over  $\operatorname{Spec} K$ . Noncommutative analogues of vector bundles were defined and used by Van den Bergh [9] to construct noncommutative  $\mathbb{P}^1$ -bundles over commutative schemes.

The purpose of this paper is to study the structure of two-sided vector spaces over  $K$  when  $K$  is a perfect field. In particular, we classify irreducible two-sided vector spaces which are finite-dimensional as ordinary  $K$ -vector spaces. We then use our classification to determine the algebraic  $K$ -theory of the category of all such two-sided vector spaces. We also give canonical representations for certain two-sided vector spaces, generalizing [5, Theorem 1.3].

The structure theory of two-sided vector spaces has important applications to noncommutative algebraic geometry via the theory of noncommutative vector bundles. Let  $S$  and  $X$  be commutative schemes and suppose  $X$  is an  $S$ -scheme of finite type. By an “ $S$ -central noncommutative vector bundle over  $X$ ” we mean an  $\mathcal{O}_S$ -central, coherent sheaf  $X$ -bimodule which is locally free on the right and left [9, Definition 2.3, p. 440]. When  $S = \operatorname{Spec} k$  and  $X = \operatorname{Spec} K$ , a sheaf  $X$ -bimodule which is locally free of finite rank on each side is nothing more than a two-sided  $K$ -vector space  $V$ , finite-dimensional on each side, where the left and right actions of  $K$  on  $V$  may differ.

When  $X$  is an integral scheme, any noncommutative vector bundle  $\mathcal{E}$  over  $X$  localizes to a noncommutative vector bundle  $\mathcal{E}_\eta$  over the generic point  $\eta$  of  $X$ . If  $\mathcal{O}_X$  acts centrally on  $\mathcal{E}$ , then  $\mathcal{E}_\eta$  is completely characterized by its dimension over the field of fractions,  $k(X)$ , of  $X$ . In this case, the rank of  $\mathcal{E}$  is defined as  $\dim_{k(X)} \mathcal{E}_\eta$ . Since localization is exact, localization induces a map  $K_0(X) \rightarrow K_0(\operatorname{Spec} k(X))$ , and the rank of  $\mathcal{E}$  can also be defined as the image of the class of  $\mathcal{E}$  via this map.

Now suppose  $X$  is of finite type over  $\operatorname{Spec} k$ . If  $\mathcal{O}_X$  does not act centrally on  $\mathcal{E}$ , then  $\mathcal{E}_\eta$  will be a two-sided vector space over  $k(X)$  whose left and right actions differ. In this case,  $\mathcal{E}_\eta$  is not completely characterized by its left and right dimension. However, localization induces a map  $K_0^B(X) \rightarrow K_0^B(\operatorname{Spec} k(X))$  where  $K_0^B(X)$  denotes the Quillen  $K$ -theory of the category of  $k$ -central noncommutative vector bundles over  $X$  and  $K_0^B(\operatorname{Spec} k(X))$  is defined similarly. It is thus reasonable to define the rank of  $\mathcal{E}$  as the image of the class of  $\mathcal{E}$  via this map. If this notion of rank is to be useful we must be able to compute the group  $K_0^B(\operatorname{Spec} k(X))$ .

In addition, one can often construct a noncommutative symmetric algebra  $\mathcal{A}$  from a noncommutative vector bundle  $\mathcal{E}$  [5, Section 2], [8, Section 5.1]. While  $\mathcal{A}$  is not generally a sheaf of algebras over  $X$ , its localization at the generic point  $\eta$  of  $X$ ,  $\mathcal{A}_\eta$ , is an algebra. The birational class of the projective bundle associated to  $\mathcal{A}$  is determined by the degree zero component of the skew field of fractions of  $\mathcal{A}_\eta$ . Since  $\mathcal{A}_\eta$  is generated by  $\mathcal{E}_\eta$ , we see that the birational class of a noncommutative projectivization is governed by a noncommutative vector bundle over  $\operatorname{Spec} K(X)$ .

We now summarize the contents of the paper. In Section 2 we describe some general properties of two-sided vector spaces that we will use in the sequel. In Section 3 we study simple objects in  $\mathbf{Vect}(K)$ , the category of two-sided  $K$ -vector spaces which are left finite-dimensional. In particular, we parameterize isomorphism classes of simple two-sided vector spaces by orbits of embeddings  $\lambda : K \rightarrow \bar{K}$  under the action of left-composition by elements of  $\operatorname{Aut}(\bar{K}/K)$  (Theorem 3.2). In Section 4, we use results from Section 3 to explicitly describe the Quillen  $K$ -groups of

$\text{Vect}(K)$ , denoted  $K_0^B(K)$  (Theorem 4.1), and give a procedure for calculating the ring structure on  $K_0^B(K)$ .

Finally in Section 5, we study matrix representations of two-sided vector spaces, i.e. homomorphisms  $\phi : K \rightarrow M_n(K)$ . Specifically, we consider the problem of finding a  $P \in GL_n(K)$  such that the homomorphism  $P\phi P^{-1}$  has a particularly nice form. We prove that if every matrix in  $\text{im } \phi$  has all of its eigenvalues in  $K$ , then the triangularized form of  $\phi$  can be described in terms of higher derivations on  $K$  (Theorem 5.4). We also develop sufficient conditions on a matrix  $A$  to ensure the existence of an upper triangular matrix  $P \in GL_n(K)$  with  $PAP^{-1}$  in Jordan canonical form (Theorem 5.8). Combining these results, we give sufficient conditions that enable us to describe the off diagonal blocks of  $P\phi P^{-1}$  (Corollary 5.10).

Throughout the paper, we provide examples of our results. We reproduce and extend the third case of [5, Theorem 1.3] by describing the structure of 2 and 3-dimensional simple two-sided vector spaces when they exist. When  $p \geq 3$  is prime and  $K = \mathbb{Q}(\sqrt[p]{2})$ , we describe the isomorphism classes of  $\mathbb{Q}$ -central two-sided  $K$ -vector spaces. There are only two, with dimensions 1 and  $p - 1$ . We then describe the ring  $K_0^B(K)$  via generators and relations. Finally, we provide an example in Section 5 to show that there exists a field  $K$ , a homomorphism  $\phi : K \rightarrow M_3(K)$ , and an element  $y \in K$  such that there is no  $P \in GL_3(K)$  with  $P\phi P^{-1}$  upper triangular and  $P\phi(y)P^{-1}$  in Jordan canonical form (Example 5.5).

## 2. Preliminaries

As we mentioned above,  $K$  will always denote a perfect field of arbitrary characteristic and  $\bar{K}$  will be a fixed algebraic closure of  $K$ . By a *two-sided vector space* we mean a  $K$ -bimodule  $V$  where the left and right actions of  $K$  on  $V$  do not necessarily coincide. Except when explicitly stated to the contrary, we shall only consider those two-sided vector spaces whose left dimension is finite, and we use the phrases “two-sided vector space” and “bimodule” interchangeably.

Since we shall only consider bimodules  $V$  with  ${}_K V$  and  $V_K$  both unital, it is easy to see that the prime subfield of  $K$  must act centrally on any two-sided vector space. We shall fix a base field  $k \subset K$  and consider only those bimodules  $V$  which are centralized by  $k$ . Note that we do not assume that  $K/k$  is algebraic in general. While all of the notions that we introduce in this paper will depend on the centralizing subfield  $k$ , it turns out that  $k$  itself will usually not play an important role in any of our results. In particular we will omit  $k$  from our notation.

Given a  $K$ -bimodule  $V$  and a set of vectors  $\{v_i : i \in I\}$ , we shall always write  $\text{span}\{v_i\}$  to stand for the *left* span of the  $v_i$ . In general,  $\text{span}\{v_i\}$  will not be a sub-bimodule of  $V$ .

If  $V$  is a two-sided vector space, then right multiplication by  $x \in K$  defines an endomorphism  $\phi(x)$  of  ${}_K V$ , and the right action of  $K$  on  $V$  is via the  $k$ -algebra homomorphism  $\phi : K \rightarrow \text{End}({}_K V)$ . This observation motivates the following definition.

**Definition 2.1.** Let  $\phi : K \rightarrow M_n(K)$  be a nonzero homomorphism. Then we denote by  ${}_1 K_\phi^n$  the two-sided vector space of left dimension  $n$ , where the left action is the usual one and the right action is via  $\phi$ ; that is,

$$x \cdot (v_1, \dots, v_n) = (xv_1, \dots, xv_n), \quad (v_1, \dots, v_n) \cdot x = (v_1, \dots, v_n)\phi(x). \quad (1)$$

We shall always write scalars as acting to the left of elements of  ${}_1 K_\phi^n$  and matrices acting to the right; thus, elements of  $K^n$  are written as row vectors and if  $v \in K^n$  is an eigenvector for  $\phi(x)$  with eigenvalue  $\lambda$ , we write  $v\phi(x) = \lambda v$ .

It is easy to see that, if  $V$  is a two-sided vector space and  $[K : k] < \infty$ , then  $\dim_K V$  is finite if and only if  $\dim V_K$  is finite, and in this case the two dimensions must be equal. Thus, when  $[K : k] < \infty$ , we may drop subscripts and simply write  $\dim V$  for this common dimension. If  $[K : k]$  is infinite, it is no longer true that the finiteness of  $\dim_K V$  implies the finiteness of  $\dim V_K$ , as the following example shows.

**Example 2.2.** Let  $K = k(x_1, x_2, \dots)$ , let  $\phi : K \rightarrow K$  be the homomorphism defined by  $\phi(x_i) = x_{i+1}$  and let  $V = {}_1K_\phi$ . Then the dimension of  ${}_K V$  is 1, while the dimension of  $V_K$  is infinite.

We denote the category of left finite-dimensional two-sided vector spaces by  $\mathbf{Vect}(K)$ . Clearly  $\mathbf{Vect}(K)$  is a finite-length category. If we write  $K^e = K \otimes_k K$  for the enveloping algebra of  $K$ , then there is a category equivalence between (not necessarily finite-dimensional)  $K$ -bimodules and (say) left  $K^e$ -modules. Under this equivalence,  $\mathbf{Vect}(K)$  can be identified as a full subcategory of the category of finite-length  $K^e$ -modules. If  $[K : k]$  is finite, then  $\mathbf{Vect}(K) = K^e\text{-mod}$ , the category of noetherian left  $K^e$ -modules. When  $K/k$  is infinite, this need no longer hold: if we define  $V = {}_\phi K_1$  in the obvious way for the map  $\phi$  in Example 2.2, then  $V$  is clearly simple in  $K^e\text{-Mod}$  but is not in  $\mathbf{Vect}(K)$ .

If  $V \in \mathbf{Vect}(K)$  with left dimension equal to  $n$ , then choosing a left basis for  $V$  shows that  $V \cong {}_1K_\phi^n$  for some homomorphism  $\phi : K \rightarrow M_n(K)$ ; we shall say that  $\phi$  represents  $V$  in this case.

If  $L$  is an extension field of  $K$ , then of course any matrix over  $K$  can be viewed as a matrix over  $L$ , and a function  $\phi : K \rightarrow M_n(K)$  can be viewed as having its image in  $M_n(L)$ . If  $A, B \in M_n(K)$ , then we write  $A \sim_L B$  if  $A$  and  $B$  are similar in  $M_n(L)$ ; that is, if  $B = PAP^{-1}$  for some  $P \in GL_n(L)$ . Similarly, if  $\phi : K \rightarrow M_n(K)$  and  $\psi : K \rightarrow M_n(K)$  are functions, we write  $\phi \sim_L \psi$  if  $\phi(x) = P\psi(x)P^{-1}$  for some  $P \in GL_n(L)$ . In either case, if  $P$  actually lives in  $M_n(K)$ , then we simply write  $\sim$  for  $\sim_K$ .

The following well known result follows readily from the fact that a homomorphism  $\phi : K \rightarrow M_n(K)$  restricts to a representation of the group  $K^*$  of units of  $K$ .

**Lemma 2.3.** *Let  $L$  be an extension field of  $K$ .  $L \otimes_K {}_1K_\phi^n \cong L \otimes_K {}_1K_\psi^n$  as  $L \otimes_K K^e$ -modules if and only if  $\phi \sim_L \psi$ .*

The next result is a special case of the Noether-Deuring Theorem [1, Exercise 6, p. 139].

**Lemma 2.4.** *Let  $L$  be an extension field of  $K$ , and let  $A, B \in M_n(K)$ . If  $A \sim_L B$ , then  $A \sim B$ . Similarly, if  $\phi : K \rightarrow M_n(K)$  and  $\psi : K \rightarrow M_n(K)$  are functions with  $\phi \sim_L \psi$ , then  $\phi \sim \psi$ .*

### 3. Simple two-sided vector spaces

The main result of this section is a determination of all of the isomorphism classes of simple two-sided vector spaces. In order to state our classification, we introduce some notation. We write  $\text{Emb}(K)$  for the set of  $k$ -embeddings of  $K$  into  $\bar{K}$ , and  $G = G(K)$  for the absolute Galois group  $\text{Aut}(\bar{K}/K)$ . (Note that  $\bar{K}/K$  is Galois since  $K$  is perfect.) If  $L$  is an intermediate field, then we write  $G(L)$  for  $\text{Aut}(\bar{K}/L)$ .

Now,  $G$  acts on  $\text{Emb}(K)$  by left composition. Given  $\lambda \in \text{Emb}(K)$ , we denote the orbit of  $\lambda$  under this action by  $\lambda^G$ , and we write  $K(\lambda)$  for the composite field  $K \vee \text{im}(\lambda)$ . The stabilizer  $G_\lambda$

of  $\lambda$  under this action is easy to calculate:  $\sigma\lambda = \lambda$  if and only if  $\sigma$  fixes  $\text{im}(\lambda)$ ; since  $\sigma$  fixes  $K$  as well we have that  $G_\lambda = G(K(\lambda))$ .

**Lemma 3.1.**  *$[K(\lambda) : K]$  is finite if and only if  $|\lambda^G|$  is finite, and in this case  $|\lambda^G| = [K(\lambda) : K]$ .*

**Proof.** By the above, the stabilizer of  $\lambda$  is  $G(K(\lambda))$ . Thus  $|\lambda^G| = [G : G(K(\lambda))]$ . The result now follows by basic Galois Theory.  $\square$

It turns out that we will only be interested in those embeddings  $\lambda$  with  $\lambda^G$  finite; we denote the set of finite orbits of  $\text{Emb}(K)$  under the action of  $G$  by  $\Lambda(K)$ . The following theorem gives our classification of simple bimodules.

**Theorem 3.2.** *There is a one-to-one correspondence between isomorphism classes of simples in  $\text{Vect}(K)$  and  $\Lambda(K)$ . Moreover, if  $V$  is a simple two-sided vector space corresponding to  $\lambda^G \in \Lambda(K)$ , then  $\dim_K V = |\lambda^G|$  and  $\text{End}(V) \cong K(\lambda)$ .*

To prove the first part of Theorem 3.2, we construct a map from the collection of simple bimodules to  $\Lambda(K)$  and show that it gives the desired bijection. We begin in greater generality, starting with a (not necessarily simple) two-sided vector space  $V$  with  $V \cong {}_1 K_\phi^n$ . Now,  $\text{im } \phi$  is a set of pairwise commuting matrices in  $M_n(K)$ ; viewing  $\text{im } \phi$  as a subset of  $M_n(\bar{K})$ , we know that there exists a common eigenvector  $v \in \bar{K}^n$  for  $\text{im } \phi$ . Define a function  $\lambda : K \rightarrow \bar{K}$  by letting  $\lambda(x)$  be the eigenvalue of  $\phi(x)$  corresponding to  $v$ ; i.e.  $v\phi(x) = \lambda(x)v$ . It is easy to check that  $\lambda$  is an embedding of  $K$  into  $\bar{K}$ , and since  $\phi$  is a  $k$ -algebra homomorphism we have that  $\lambda \in \text{Emb}(K)$ .

**Lemma 3.3.** *If  $v \in \bar{K}^n$  is a common eigenvector for  $\text{im } \phi$  with corresponding eigenvalue  $\lambda$ , then  $\lambda \in \Lambda(K)$ . Moreover,  $|\lambda^G| \leq n$ .*

**Proof.** Note first that if  $\sigma \in G$ ,  $\sigma(v)$  is also a common eigenvector of  $\text{im } \phi$ , with corresponding eigenvalue  $\sigma\lambda$ . Indeed, we compute

$$\sigma(v)\phi(x) = \sigma(v)\sigma(\phi(x)) = \sigma(v\phi(x)) = \sigma(\lambda(x)v) = \sigma\lambda(x)\sigma(v). \quad (2)$$

Now, if  $\sigma\lambda \neq \tau\lambda$ , then for at least one value of  $x \in K$  the vectors  $\sigma(v)$  and  $\tau(v)$  are eigenvectors for  $\phi(x)$  with different eigenvalues; from this it follows that  $\sigma(v)$  and  $\tau(v)$  are linearly independent. If  $\lambda^G = \{\sigma_i\lambda : i \in I\}$ , then  $\{\sigma_i(v) : i \in I\}$  is a linearly independent subset of  $\bar{K}^n$ . Thus  $|\lambda^G| \leq n$  and in particular  $\lambda^G \in \Lambda(K)$ .  $\square$

Viewing  $\lambda$  as an embedding of  $K$  into  $K(\lambda)$ , we may without loss of generality assume that the common eigenvector  $v$  for  $\text{im } \phi$  with eigenvalue  $\lambda$  lives in  $K(\lambda)^n$ . We now fix notation which will be useful when proving Theorem 3.2. We let  $m = [K(\lambda) : K] = |\lambda^G|$  and we fix a basis  $\{\alpha_1, \dots, \alpha_m\}$  for  $K(\lambda)/K$ . We may write

$$v = \sum_{i=1}^m \alpha_i v_i \quad (3)$$

with each  $v_i \in K^n$  and

$$\lambda(x) = \sum_{i=1}^m \lambda_i(x) \alpha_i,$$

where each  $\lambda_i : K \rightarrow K$  is an additive function. Finally, we let  $\beta_{ijk}$  denote the structure constants for the basis  $\{\alpha_1, \dots, \alpha_m\}$ ; that is,

$$\alpha_i \alpha_j = \sum_{k=1}^m \beta_{ijk} \alpha_k.$$

**Lemma 3.4.** *In the above notation,  $\text{span}\{v_1, \dots, v_m\}$  is a two-sided subspace of  $V$ . In particular, if  $V$  is simple,  $\dim_K V = |\lambda^G|$ .*

**Proof.** We must show that  $v_i \phi(x) \in \text{span}\{v_1, \dots, v_m\}$  for all  $x \in K$  and all  $i$ . On the one hand,  $v \phi(x) = (\sum_i \alpha_i v_i) \phi(x) = \sum_i \alpha_i v_i \phi(x)$ . On the other hand,

$$\begin{aligned} v \phi(x) &= \lambda(x) v = \left( \sum_p \lambda_p(x) \alpha_p \right) \left( \sum_q \alpha_q v_q \right) \\ &= \sum_{p,q} \lambda_p(x) \alpha_p \alpha_q v_q = \sum_i \alpha_i \left( \sum_{p,q} \beta_{pqi} \lambda_p(x) v_q \right). \end{aligned} \quad (4)$$

Matching up coefficients of  $\alpha_i$  shows that  $v_i \phi(x) = \sum_{p,q} \beta_{pqi} \lambda_p(x) v_q$ , so that  $v_i \phi(x) \in \text{span}\{v_1, \dots, v_m\}$ . This proves the first assertion.

If  $V$  is simple, the first part of the lemma implies  $V = \text{span}\{v_1, \dots, v_m\}$ . Thus,  $m = |\lambda^G| \geq \dim_K V$ . On the other hand,  $|\lambda^G| \leq \dim_K V$  by Lemma 3.3. Thus,  $|\lambda^G| = \dim_K V$  when  $V$  is simple.  $\square$

**Proposition 3.5.** *Let  $\phi : K \rightarrow M_n(K)$  be a homomorphism and let  $\lambda : K \rightarrow \bar{K}$  be the eigenvalue of a common eigenvector of  $\text{im } \phi \subset M_n(\bar{K})$ . The map*

$$\Phi : \{\text{Isomorphism classes of simples in } \mathbf{Vect}(K)\} \rightarrow \Lambda(K)$$

*defined by  $\Phi([{}_1 K_\phi^n]) = \lambda^G$  is a bijection.*

**Proof.** *Part 1.* We show  $\Phi$  is an injection.

*Part 1, Step 1.* We show  $\Phi$  is well defined. Let  $V$  be a simple object in  $\mathbf{Vect}(K)$ , and suppose  $V \cong {}_1 K_\phi^n$ . By Lemma 3.4,  $|\lambda^G| = n$ . Let us write out the elements of  $\lambda^G$  as  $\{\lambda, \sigma_2 \lambda, \dots, \sigma_n \lambda\}$ . Then taking  $\{v, \sigma_2(v), \dots, \sigma_n(v)\}$  as a basis for  $\bar{K}^n$ , we see that there exists  $Q \in GL_n(\bar{K})$  such that

$$Q \phi(x) Q^{-1} = \text{diag}(\lambda(x), \sigma_2 \lambda(x), \dots, \sigma_n \lambda(x)) \quad (5)$$

for all  $x \in K$ . In particular, if  $\mu : K \rightarrow \bar{K}$  is the eigenvalue for  $\phi(x)$  corresponding to some common eigenvector  $w$  of  $\text{im } \phi$ , then we must have  $\mu = \sigma_i \lambda$  for some  $i$ ; that is,  $\mu^G = \lambda^G$ .

If we choose a different isomorphism  $V \cong {}_1 K_\psi^n$ , then  $\phi \sim \psi$ ; say  $\phi \cong P \psi P^{-1}$  for some  $P \in GL_n(K)$ . If  $v$  is a common eigenvector for  $\text{im } \phi$  with corresponding eigenvalue  $\lambda$ , then an easy computation shows that  $vP$  is a common eigenvector for  $\text{im}(\psi)$  with corresponding eigenvalue  $\lambda$ .

*Part 1, Step 2.* We show  $\Phi$  is an injection. If  $K$  is finite, then every embedding of  $K$  into  $\bar{K}$  is in fact an automorphism of  $K$ . Hence every simple in  $\mathbf{Vect}(K)$  is isomorphic to  ${}_1 K_\phi$  for some  $\phi \in \text{Aut}(K)$ , and the above correspondence just sends  ${}_1 K_\phi$  to  $\phi$ . Thus the claim follows when  $K$  is finite.

Now suppose that  $K$  is infinite,  $\Phi([V]) = \lambda^G = \Phi([W])$  and  $|\lambda^G| = n$ . Write  $V \cong {}_1K_\phi^n$  and  $W \cong {}_1K_\psi^n$ . As in Eq. (5), there are invertible matrices  $P, Q \in M_n(\bar{K})$  such that

$$P\phi(x)P^{-1} = Q\psi(x)Q^{-1} = \text{diag}(\lambda(x), \sigma_2\lambda(x), \dots, \sigma_n\lambda(x)), \quad (6)$$

so that  $\phi \sim_{\bar{K}} \psi$ . By Lemma 2.4,  $\phi \sim \psi$  and  $V \cong W$ .

*Part 2.* Let  $\lambda : K \rightarrow \bar{K}$  be an embedding with  $\lambda^G \in \Lambda(K)$ . We shall construct a simple two-sided vector space  $V(\lambda) = {}_1K_\phi^n$  from  $\lambda$ , such that  $v = (\alpha_1, \dots, \alpha_n) \in K(\lambda)^n$  is a common eigenvector for  $\text{im } \phi$ , with corresponding eigenvalue  $\lambda$ . Retaining the above notation, we define a map  $\phi = (\phi_{ij}) : K \rightarrow M_n(K)$  by

$$\phi_{ij}(x) = \sum_{k=1}^n \beta_{jki} \lambda_k(x). \quad (7)$$

*Part 2, Step 1.* We prove that, for all  $\sigma \in G$  and  $x \in K$ ,  $\sigma(v)$  is an eigenvector for  $\phi(x)$  with eigenvalue  $\sigma\lambda(x)$ . We have  $\sigma(v) = (\sigma(\alpha_1), \dots, \sigma(\alpha_n))$  and  $\sigma\lambda(x) = \sum_{i=1}^n \lambda_i(x)\sigma(\alpha_i)$ . On the one hand,

$$\begin{aligned} \sigma(v)\phi(x) &= (\sigma(\alpha_1), \dots, \sigma(\alpha_n))\phi(x) \\ &= \left( \sum_i \phi_{i1}(x)\sigma(\alpha_i), \dots, \sum_i \phi_{in}(x)\sigma(\alpha_i) \right) \\ &= \left( \sum_{i,k} \beta_{1ki} \lambda_k(x)\sigma(\alpha_i), \dots, \sum_{i,k} \beta_{nki} \lambda_k(x)\sigma(\alpha_i) \right). \end{aligned} \quad (8)$$

On the other hand,

$$\begin{aligned} \sigma\lambda(x)\sigma(v) &= \left( \sum_k \lambda_k(x)\sigma(\alpha_k) \right) (\sigma(\alpha_1), \dots, \sigma(\alpha_n)) \\ &= \left( \sum_k \lambda_k(x)\sigma(\alpha_k\alpha_1), \dots, \sum_k \lambda_k(x)\sigma(\alpha_k\alpha_n) \right) \\ &= \left( \sum_{i,k} \lambda_k(x)\beta_{k1i}\sigma(\alpha_i), \dots, \sum_{i,k} \lambda_k(x)\beta_{kni}\sigma(\alpha_i) \right). \end{aligned} \quad (9)$$

Comparing coordinates and using the identity  $\beta_{pqr} = \beta_{qpr}$  for all  $p, q, r$  gives the result.

*Part 2, Step 2.* We show  $\phi$  is a homomorphism. Since each  $\lambda_k$  is an additive function it is clear that  $\phi$  is additive. To see that  $\phi$  is multiplicative, write out  $\lambda^G = \{\sigma_1\lambda, \dots, \sigma_n\lambda\}$  (where  $\sigma_1$  is the identity). Then  $\{\sigma_1(v), \dots, \sigma_n(v)\}$  is a basis for  $\bar{K}^n$ , and for all  $x, y \in K$ , we have

$$\begin{aligned} \sigma_i(v)\phi(x)\phi(y) &= \sigma_i\lambda(x)\sigma_i(v)\phi(y) = \sigma_i\lambda(x)\sigma_i\lambda(y)\sigma_i(v) \\ &= \sigma_i\lambda(xy)\sigma_i(v) = \sigma_i(v)\phi(xy). \end{aligned} \quad (10)$$

This shows that  $\phi(x)\phi(y)$  and  $\phi(xy)$  act as the same linear transformation on each  $\sigma_i(v)$ . Since the  $\sigma_i(v)$  form a basis for  $\bar{K}^n$ , we have that  $\phi(x)\phi(y) = \phi(xy)$  for all  $x, y \in K$ .

*Part 2, Step 3.* Since  $\phi$  is a homomorphism, we can define the two-sided vector space  $V(\lambda) = {}_1K_\phi^n$ . We prove  $V(\lambda)$  is simple. Suppose that  $W$  is a simple sub-bimodule of  $V(\lambda)$  with  $\dim W = m$ , and fix a left basis for  $V(\lambda)$  containing a left basis for  $W$ . Then, relative to this basis, we have  $V(\lambda) \cong {}_1K_\psi^n$ , where  $\psi = \begin{pmatrix} \psi_1 & \theta \\ 0 & \psi_2 \end{pmatrix}$  and  $W \cong {}_1K_{\psi_2}^m$ . Since  $W$  is simple, there is a unique orbit  $\mu^G = \{\mu_1, \dots, \mu_m\} \in A(K)$  with  $\psi_2 \sim_{\bar{K}} \text{diag}(\mu_1, \dots, \mu_m)$ . On the other hand, we have by definition of  $V(\lambda)$  that  $\phi \sim_{\bar{K}} \text{diag}(\lambda, \sigma_2\lambda, \dots, \sigma_n\lambda)$ ; since  $\phi \sim \psi$  we see that  $\mu_1 = \sigma_j\lambda$  for some  $j$ . Hence  $\mu^G = \lambda^G$  and  $W = V(\lambda)$  since  $\Phi$  is injective.  $\square$

To complete the proof of Theorem 3.2, we need to compute  $\text{End}(V(\lambda))$ .

**Proposition 3.6.**  $\text{End}(V(\lambda)) \cong K(\lambda)$ .

**Proof.** Let  $|\lambda^G| = n$ . We first note that  $\text{End}(V(\lambda))$  can be made into a left vector space over  $K$  by defining  $(xf)(v) = xf(v)$  for  $x \in K$ ,  $v \in V(\lambda)$ , and  $f \in \text{End}(V(\lambda))$ . Also, since  $V(\lambda)$  is a simple bimodule, it is generated as a bimodule by a single element  $w$ . If  $\{f_1, \dots, f_{n+1}\}$  is a subset of  $\text{End}(V(\lambda))$ , then  $\{f_1(w), \dots, f_{n+1}(w)\}$  are necessarily linearly dependent in  $V(\lambda)$ ; hence there exist  $x_i \in K$  such that the endomorphism  $\sum_{i=1}^{n+1} x_i f_i$  acts as 0 on  $w$ . Since  $w$  generates  $V(\lambda)$  we see that  $\sum_{i=1}^{n+1} x_i f_i = 0$  and so  $\dim \text{End}(V(\lambda)) \leq n$ .

Fix an isomorphism  $V(\lambda) \cong {}_1K_\phi^n$ , and let  $\{e_1, \dots, e_n\}$  be the standard basis for  $K^n$ . Given  $f \in \text{End}(V(\lambda))$ , we can write  $f(e_i) = \sum_j f_{ji} e_j$ , where  $f_{ji} \in K$ . Then the map  $f \mapsto M(f) = (f_{ij})$  allows us to realize each  $f \in \text{End}(V(\lambda))$  as right multiplication by the matrix  $M(f) \in M_n(K)$ . The fact that  $f$  is a bimodule endomorphism is equivalent to  $M(f)$  commuting with  $\phi(x)$  for all  $x \in K$ . Conversely, if  $M \in M_n(K)$  with  $M = (m_{ij})$  and if  $M$  commutes with  $\phi(x)$ , the rule  $e_i \mapsto \sum_j m_{ji} e_j$  makes  $M$  an element of  $\text{End}(V(\lambda))$ .

For each  $p \leq n$ , let  $M(p)$  be the matrix given by  $M(p)_{ij} = \beta_{pji}$ . We prove that  $M(p) \in \text{End}(V(\lambda))$ . If  $v = (\alpha_1, \dots, \alpha_n)$  as in (3), then one calculates that

$$\sigma(v)M(p) = (\sigma(\alpha_1), \dots, \sigma(\alpha_n))(\beta_{pji}) = \left( \sum_j \beta_{p1j} \sigma(\alpha_j), \dots, \sum_j \beta_{pnj} \sigma(\alpha_j) \right) \quad (11)$$

for all  $\sigma \in \text{Aut}(\bar{K}/K)$ . On the other hand,

$$\sigma(\alpha_p) \sigma(\alpha_i) = \sigma(\alpha_p \alpha_i) = \sigma \left( \sum_j \beta_{pij} \alpha_j \right) = \sum_j \beta_{pij} \sigma(\alpha_j). \quad (12)$$

Hence we see that the  $i$ th component of  $\sigma(v)M(p)$  is  $\sigma(\alpha_p) \sigma(\alpha_i)$ , and we conclude that  $\sigma(v)$  is an eigenvector for  $M(p)$  with eigenvalue  $\sigma(\alpha_p)$ ; in particular, we see that  $\sigma(v)M(p)M(q) = \sigma(v)M(q)M(p)$  for all  $p, q \leq n$  and  $\sigma \in \text{Aut}(\bar{K}/K)$ . Since  $\{v, \sigma_2(v), \dots, \sigma_n(v)\}$  is a basis for  $\bar{K}^n$ , we conclude that in fact  $M(p)$  and  $M(q)$  commute for all  $p, q$ . Finally, since  $\sigma_i(v)$  is a common eigenvector for  $\phi(x)$  and  $M(p)$  for all  $p \leq n$  and  $x \in K$ , we see that  $M(p)$  and  $\phi(x)$  commute. Therefore,  $\{M(1), \dots, M(n)\}$  are pairwise commuting,  $K$ -linearly independent elements of  $\text{End}(V(\lambda))$ . Since  $\dim \text{End}(V(\lambda)) \leq n$ , we conclude that  $\text{End}(V(\lambda)) \cong K\{M(1), \dots, M(n)\}$ ; one checks easily that the map  $M(p) \mapsto \alpha_p$  gives the desired ring isomorphism  $\text{End}(V(\lambda)) \cong K(\lambda)$ .  $\square$



We illustrate Theorem 3.2 with several examples.

**Example 3.7.** Suppose that there exists  $\lambda \in \text{Emb}(K)$  with  $|\lambda^G| = 2$ . Then  $K(\lambda)$  is a degree 2 extension of  $K$ , and so  $K(\lambda) = K(\sqrt{m})$  for some  $m \in K$ . Then  $\text{Aut}(K(\sqrt{m})/K)$  is generated by  $\sigma$ , where  $\sigma(\sqrt{m}) = -\sqrt{m}$ , and  $\lambda^G = \{\lambda, \sigma\lambda\}$ .

Using  $\{1, \sqrt{m}\}$  as a  $K$ -basis for  $\sqrt{m}$ , we can write  $\lambda(x) = \lambda_1(x) + \lambda_2(x)\sqrt{m}$ . If we write out the matrix  $(\phi_{ij}(x))$ , we see that

$$\phi(x) = \begin{pmatrix} \lambda_1(x) & m\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) \end{pmatrix}, \quad (13)$$

and that

$$(1, \sqrt{m}) \begin{pmatrix} \lambda_1(x) & m\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) \end{pmatrix} = (\lambda(x), \lambda(x)\sqrt{m}) = \lambda(x) (1, \sqrt{m}).$$

Moreover, the fact that  $\phi$  is a homomorphism gives the formulas

$$\begin{aligned} \lambda_1(xy) &= \lambda_1(x)\lambda_1(y) + m\lambda_2(x)\lambda_2(y), \\ \lambda_2(xy) &= \lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x). \end{aligned}$$

Thus we recover [5, Theorem 1.3(iii)] as a special case of Theorem 3.2.

**Example 3.8.** Let  $K$  be a field of characteristic different from 3, and suppose there exists  $\lambda \in \text{Emb}(K)$  such that  $|\lambda^G| = 3$ . Then  $[K(\lambda) : K] = 3$ , so that  $K(\lambda) = K(\gamma)$ , where  $\gamma$  is the root of an irreducible polynomial  $x^3 + bx + c$  with  $b, c \in K$ . Thus,  $\{1, \gamma, \gamma^2\}$  is a basis of  $K(\lambda)/K$ . In this basis, we find that  $(\phi_{ij}(x))$  is the matrix

$$\begin{pmatrix} \lambda_0(x) & -c\lambda_2(x) & -c\lambda_1(x) \\ \lambda_1(x) & \lambda_0(x) - b\lambda_2(x) & -b\lambda_1(x) - c\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) & \lambda_0(x) - b\lambda_2(x) \end{pmatrix},$$

where the functions  $\lambda_0, \lambda_1$  and  $\lambda_2$  satisfy the relations

$$\begin{aligned} \lambda_0(xy) &= \lambda_0(x)\lambda_0(y) - c\lambda_2(x)\lambda_1(y) - c\lambda_1(x)\lambda_2(y), \\ \lambda_1(xy) &= \lambda_1(x)\lambda_0(y) + (\lambda_0(x) + b\lambda_2(x))\lambda_1(y) - (b\lambda_1(x) + c\lambda_2(x))\lambda_2(y), \\ \lambda_2(xy) &= \lambda_2(x)\lambda_0(y) + \lambda_1(x)\lambda_1(y) + (\lambda_0(x) - b\lambda_2(x))\lambda_2(y). \end{aligned}$$

**Example 3.9.** Suppose  $p \geq 3$  is prime,  $\rho = \sqrt[p]{2}$ ,  $\zeta$  is a primitive  $p$ th root of unity,  $k = \mathbb{Q}$  and  $K = \mathbb{Q}(\rho)$ . Then  $K(\zeta)$  is the Galois closure of  $K/\mathbb{Q}$ , with  $\text{Aut}(K(\zeta)/K) = \{\sigma_i : 1 \leq i \leq p-1\}$ , where  $\sigma_i(\zeta) = \zeta^i$ . If we let  $\lambda : K \rightarrow \bar{K}$  be the embedding that takes  $\rho$  to  $\zeta\rho$ , then  $\text{Emb}(K) = \{\text{Id}_K\} \cup \{\sigma_i\lambda : 1 \leq i \leq p-1\}$ . Hence  $\Lambda(K)$  consists of the two orbits  $\{\text{Id}_K\}$  and  $\lambda^G = \{\sigma_i\lambda : 1 \leq i \leq p-1\}$ , and so there are up to isomorphism two simples in  $\text{Vect}(K)$ : the trivial simple bimodule  $K$  corresponding to  $\{\text{Id}_K\}$ , and a  $p-1$ -dimensional simple corresponding to  $\lambda^G$ .

We now construct the matrix homomorphism  $\phi : K \rightarrow M_{p-1}(K)$  representing the  $p-1$ -dimensional simple as in (7). First, taking  $\{1, \zeta, \dots, \zeta^{p-2}\}$  as a basis of  $K(\zeta)/K$  and letting  $\alpha_i = \zeta^i$  for  $0 \leq i \leq p-2$  (we have shifted our indices for ease of computation), we compute the constants  $\beta_{jki}$ : if  $j+k \neq p-1$ , then

$$\alpha_j\alpha_k = \zeta^j\zeta^k = \zeta^{j+k} = \alpha_{j+k},$$

where the superscripts and subscripts are taken modulo  $p$ . Therefore, when  $j + k \neq p - 1$ ,  $\beta_{jki} = 1$  if and only if  $i \equiv j + k \pmod{p}$ , and  $\beta_{jki} = 0$  otherwise.

If  $j + k = p - 1$ , then

$$\alpha_j \alpha_k = \zeta^{p-1} = -1 - \zeta - \dots - \zeta^{p-2} = -\alpha_0 - \alpha_1 - \dots - \alpha_{p-2}.$$

Therefore, when  $j + k = p - 1$ ,  $\beta_{jki} = -1$  for all  $0 \leq i \leq p - 2$ .

Thus,

- (1) if  $j = 0$  then  $j + k \neq p - 1$ , so  $\beta_{0ki} = \delta_{ki}$ , and
- (2) if  $j \neq 0$ , either
  - (a)  $k = p - 1 - j$ , in which case  $\beta_{j,p-1-j,i} = -1$  for all  $i$  or
  - (b)  $k \neq p - 1 - j$ , in which case  $\beta_{j,i-j,i} = 1$  for all  $i \neq j - 1$  (where subscripts are taken modulo  $p$ ) and  $\beta_{jki} = 0$  otherwise.

Next, we write

$$\lambda(x) = \lambda_0(x) + \lambda_1(x)\zeta + \dots + \lambda_{p-2}(x)\zeta^{p-2}$$

and determine the functions  $\lambda_i(x)$ ,  $0 \leq i \leq p - 2$ . If  $x \in K$ , we may write  $x = \sum_{l=0}^{p-1} a_l \rho^l$  with  $a_0, \dots, a_{p-1} \in \mathbb{Q}$ . It is then easy to see that

$$\lambda_i \left( \sum_{l=0}^{p-1} a_l \rho^l \right) = a_i \rho^i - a_{p-1} \rho^{p-1}$$

for  $0 \leq i \leq p - 2$ .

Using the formula  $\phi_{ij}(x) = \sum_{k=0}^{p-2} \beta_{jki} \lambda_k(x)$ , we may deduce that  $\phi(x)$  is the matrix

$$\begin{pmatrix} \lambda_0(x) & -\lambda_1(x) \\ \lambda_1(x) & -\lambda_1(x) + \lambda_0(x) \end{pmatrix} \quad (14)$$

when  $p = 3$  and  $\phi(x)$  is the matrix

$$\begin{pmatrix} \lambda_0(x) & -\lambda_{p-2}(x) & -\lambda_{p-3}(x) + \lambda_{p-2}(x) & \cdots & -\lambda_1(x) + \lambda_2(x) \\ \lambda_1(x) & -\lambda_{p-2}(x) + \lambda_0(x) & -\lambda_{p-3}(x) & \cdots & -\lambda_1(x) + \lambda_3(x) \\ \lambda_2(x) & -\lambda_{p-2}(x) + \lambda_1(x) & -\lambda_{p-3}(x) + \lambda_0(x) & \cdots & -\lambda_1(x) + \lambda_4(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{p-3}(x) & -\lambda_{p-2}(x) + \lambda_{p-4}(x) & -\lambda_{p-3}(x) + \lambda_{p-5}(x) & \cdots & -\lambda_1(x) \\ \lambda_{p-2}(x) & -\lambda_{p-2}(x) + \lambda_{p-3}(x) & -\lambda_{p-3} + \lambda_{p-4}(x) & \cdots & -\lambda_1(x) + \lambda_0(x) \end{pmatrix}$$

when  $p \geq 5$ .

Had we chosen a different basis for  $K(\lambda)$  over  $K$ , then  $\phi(x)$  would have a different form. For example, when  $p = 3$ , then  $K(\zeta) = K(\sqrt{-3})$ . If we use  $\{1, \sqrt{-3}\}$  as a basis for  $K(\zeta)$  over  $K$ , then we find that  $\phi(x)$  takes the form (13).

We conclude this section by noting that there are no nontrivial extensions between nonisomorphic simple bimodules. The result is probably well known, but we were unable to find a reference.

**Proposition 3.10.**  $\text{Ext}_{K^e}^1(V, W) = 0$  for nonisomorphic simple bimodules  $V, W$ .

**Proof.** Suppose that  $V$  and  $W$  are nonisomorphic simple bimodules. Let  $\Phi([V]) = \lambda^G$  and  $\Phi([W]) = \mu^G$ , and fix isomorphisms  $V \cong {}_1K_\phi^m$  and  $W \cong {}_1K_\psi^n$ . If  $U$  is an extension of  $V$  by  $W$ , then there is a basis for  $K^{m+n}$  such that  $U \cong {}_1K_\eta^{m+n}$ , where  $\eta = \begin{pmatrix} \psi & \theta \\ 0 & \phi \end{pmatrix}$  for some  $\theta : K \rightarrow M_{n \times m}(K)$ . Enumerate the elements of  $\lambda^G$  and  $\mu^G$  as  $\{\lambda_1, \dots, \lambda_m\}$  and  $\{\mu_1, \dots, \mu_n\}$ , respectively. Then we have that

$$\begin{pmatrix} \phi & \theta \\ 0 & \psi \end{pmatrix} \sim_{\bar{K}} \text{diag}(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n) \sim_{\bar{K}} \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}. \quad (15)$$

It follows by Lemma 2.4 that  $U \cong V \oplus W$ .  $\square$

#### 4. Algebraic $K$ -theory of $\mathbf{Vect}(K)$

We shall denote by  $K_i^B(K)$  the Quillen  $K$ -theory of  $\mathbf{Vect}(K)$  [6] (the superscript stands for “bimodule”). The description of the simples in  $\mathbf{Vect}(K)$  in Section 3 and the Devissage Theorem [6, Corollary 5.1] immediately yield the following result.

**Theorem 4.1.** *For all  $i \geq 0$ , there is an isomorphism of abelian groups*

$$K_i^B(K) \cong \bigoplus_{\lambda^G \in A(K)} K_i(K(\lambda)). \quad (16)$$

The Grothendieck group  $K_0^B(K)$  can be made into a commutative ring by defining multiplication via the tensor product. Thus, if  $V$  and  $W$  are simple bimodules in  $\mathbf{Vect}(K)$ , we define  $[V] \cdot [W] = [V \otimes W]$  in  $K_0^B(K)$ . (Here and below  $\otimes$  denotes the tensor product over  $K$ .) In particular,  $[V] \cdot [W] = \sum_{i=1}^t [V_i]$ , where  $V_1, \dots, V_t$  are the composition factors of  $V \otimes W$ . There is an especially nice description of  $K_0^B(K)$  when  $\text{Emb}(K) = \text{Aut}(K)$ ; this will happen for instance if  $K$  is a normal algebraic extension of the centralizing subfield  $k$ .

**Proposition 4.2.** *If  $K$  is a field with  $\text{Emb}(K) = \text{Aut}(K)$ , then there is a ring isomorphism  $K_0^B(K) \cong \mathbb{Z}[\text{Aut}(K)]$ .*

**Proof.** Each simple bimodule in  $\mathbf{Vect}(K)$  is isomorphic to  ${}_1K_\phi$  for some  $\phi \in \text{Aut}(K)$ . The map  $[{}_1K_\phi] \mapsto \phi$  then gives an isomorphism between the abelian groups  $K_0^B(K)$  and  $\mathbb{Z}[\text{Aut}(K)]$ . Moreover, an elementary calculation shows that  ${}_1K_\phi \otimes {}_1K_\psi \cong {}_1K_{\phi\psi}$ . From this it follows readily that the above map is actually a ring isomorphism.  $\square$

In order to describe the ring structure of  $K_0^B(K)$  for a general field  $K$ , we shall need to introduce some notation. Identifying the  $K$ -algebras  $M_m(K) \otimes M_n(K)$  and  $M_{mn}(K)$ , we introduce multi-index notation to refer to the coordinates of  $M_{mn}(K)$  as follows. Order the pairs  $(i, j)$  with  $1 \leq i \leq m$  and  $1 \leq j \leq n$  lexicographically; then there is a bijection between these pairs and  $\{1, \dots, mn\}$ . We shall write  $A_{(i_1, i_2), (j_1, j_2)}$  for the entry of  $A \in M_{mn}(K)$  whose row corresponds to  $(i_1, i_2)$  and whose column corresponds to  $(j_1, j_2)$  under this bijection. The reason for adopting this notation is that, if  $A = (a_{ij}) \in M_m(K)$  and  $B = (b_{ij}) \in M_n(K)$ , then  $(A \otimes B)_{(i_1, i_2), (j_1, j_2)} = a_{i_1 j_1} b_{i_2 j_2}$ , where  $A \otimes B$  is the Kronecker product of  $A$  and  $B$ .

The following is a variant of the Kronecker product for functions.

**Definition 4.3.** Let  $\phi = (\phi_{ij}) : K \rightarrow M_m(K)$  and  $\psi = (\psi_{ij}) : K \rightarrow M_n(K)$  be functions. Then we define their *Kronecker composition*  $\phi \otimes \psi : K \rightarrow M_{mn}(K)$  by the rule  $(\phi \otimes \psi)_{(i_1, i_2), (j_1, j_2)} = \phi_{i_1 j_1} \circ \psi_{i_2 j_2}$ . Similarly, if  $A = (a_{ij}) \in M_n(K)$ , then we define  $\phi \otimes A \in M_{mn}(K)$  to be the matrix given by  $(\phi \otimes A)_{(i_1, i_2), (j_1, j_2)} = \phi_{i_1 j_1}(a_{i_2 j_2})$ . Note that if  $x \in K$ , then  $(\phi \otimes \psi)(x) = \phi \otimes (\psi(x))$ , so that the two definitions are consistent with each other. Finally, if  $B \in M_m(K)$ , then we define the functions  $\phi B$  and  $B\phi$  by  $(\phi B)(x) = \phi(x)B$  and  $(B\phi)(x) = B\phi(x)$ , respectively.

The utility of the Kronecker composition in understanding tensor products of bimodules is revealed in the following lemma. In particular, it implies that when  $\phi : K \rightarrow M_m(K)$  and  $\psi : K \rightarrow M_n(K)$  are homomorphisms, so too is  $\phi \otimes \psi : K \rightarrow M_{mn}(K)$ .

**Lemma 4.4.** Given homomorphisms  $\phi : K \rightarrow M_m(K)$  and  $\psi : K \rightarrow M_n(K)$ , we have  ${}_1K_\phi^m \otimes {}_1K_\psi^n \cong {}_1K_{\phi \otimes \psi}^{mn}$ .

**Proof.** Let  $\{e_1, \dots, e_m\}$  and  $\{f_1, \dots, f_n\}$  be the standard left bases for  $K^m$  and  $K^n$ , respectively. If we let  $e_{(i,j)} = e_i \otimes f_j$ , then  $\{e_{(i,j)} : 1 \leq i \leq m, 1 \leq j \leq n\}$  gives a left basis for  $K^{mn}$ . We compute the right action of  $K$  on  $K^{mn}$  under this basis:

$$\begin{aligned} e_{(i,j)} \cdot x &= (e_i \otimes f_j) \cdot x = e_i \otimes f_j \psi(x) = e_i \otimes \sum_{l=1}^n \psi_{jl}(x) f_l \\ &= \sum_{l=1}^n e_i \cdot \psi_{jl}(x) \otimes f_l = \sum_{l=1}^n e_i \phi(\psi_{jl}(x)) \otimes f_l \\ &= \sum_{l=1}^n \sum_{k=1}^m \phi_{ik}(\psi_{jl}(x)) e_k \otimes f_l = \sum_{(k,l)} \phi_{ik} \circ \psi_{jl}(x) e_{(k,l)}. \\ &= \sum_{(k,l)} (\phi \otimes \psi)_{(i,j), (k,l)}(x) e_{(k,l)} = e_{(i,j)} (\phi \otimes \psi)(x). \quad \square \end{aligned} \tag{17}$$

**Lemma 4.5.** Let  $\phi : K \rightarrow M_m(K)$  and  $\psi : K \rightarrow M_n(K)$  be homomorphisms and let  $A = (a_{ij}) \in M_m(K)$ ,  $B = (b_{ij})$ ,  $C = (c_{ij}) \in M_n(K)$ . Then the following hold:

- (1)  $(\phi \otimes B)(\phi \otimes C) = \phi \otimes BC$ .
- (2)  $(A \otimes I_n)(\phi \otimes B) = (A\phi) \otimes B$  and  $(\phi \otimes B)(A \otimes I_n) = (\phi A) \otimes B$ , where  $I_n$  is the  $n \times n$  identity matrix.
- (3) If  $\phi \sim \phi'$  and  $\psi \sim \psi'$ , then  $\phi \otimes \psi \sim \phi' \otimes \psi'$ .

**Proof.** (1) We compute the  $(i_1, i_2), (j_1, j_2)$  component of  $(\phi \otimes B)(\phi \otimes C)$ :

$$\begin{aligned} (\phi \otimes B)(\phi \otimes C)_{(i_1, i_2), (j_1, j_2)} &= \sum_{(k,l)} (\phi \otimes B)_{(i_1, i_2), (k,l)} (\phi \otimes C)_{(k,l), (j_1, j_2)} \\ &= \sum_{(k,l)} \phi_{i_1 k}(b_{i_2 l}) \phi_{k j_1}(c_{l j_2}) \end{aligned}$$

$$\begin{aligned}
&= \sum_l \phi_{i_1 j_1} (b_{i_2 l} c_{l j_2}) \quad (\phi \text{ is a homomorphism}) \\
&= \phi_{i_1 j_1} ((BC)_{i_2 j_2}) = (\phi \otimes BC)_{(i_1, i_2), (j_1, j_2)}. \quad (18)
\end{aligned}$$

(2) Again, the proof is a computation. We show the first equality and leave the second to the reader.

$$\begin{aligned}
(A \otimes I_n)(\phi \otimes B)_{(i_1, i_2), (j_1, j_2)} &= \sum_{(k, l)} (A \otimes I_n)_{(i_1, i_2), (k, l)} (\phi \otimes B)_{(k, l), (j_1, j_2)} \\
&= \sum_{(k, l)} a_{i_1 k} (I_n)_{i_2 l} \phi_{k j_1} (b_{l j_2}). \quad (19)
\end{aligned}$$

The only nonzero term in the sum occurs when  $l = i_2$ , because of the  $(I_n)_{i_2 l}$  term. Hence the above sum collapses to

$$\sum_k a_{i_1 k} \phi_{k j_1} (b_{i_2 j_2}) = (A\phi)_{i_1 j_1} (b_{i_2 j_2}) = (A\phi \otimes B)_{(i_1, i_2), (j_1, j_2)}. \quad (20)$$

(3) First, suppose that  $B \in M_n(K)$  is invertible. Then by part (1), we have  $(\phi \otimes B)(\phi \otimes B^{-1}) = \phi \otimes BB^{-1} = \phi \otimes I_n = I_{mn}$ . Thus  $\phi \otimes B$  is invertible, with inverse  $\phi \otimes B^{-1}$ . Now, suppose that  $B\psi(x)B^{-1} = \psi'(x)$  for all  $x \in K$ , and that  $A\phi(x)A^{-1} = \phi'(x)$  for all  $x \in K$ . Then, for all  $x \in K$ , we have

$$(A \otimes I_n)(\phi \otimes B)(\phi \otimes \psi(x))(\phi \otimes B^{-1})(A^{-1} \otimes I_n) = (\phi' \otimes \psi')(x) \quad (21)$$

by parts (1) and (2) above. Hence  $\phi \otimes \psi \sim \phi' \otimes \psi'$ .  $\square$

Any embedding  $\lambda$  of  $K$  into  $\bar{K}$  can be lifted to an automorphism  $\bar{\lambda}$  of  $\bar{K}$ , such that  $\bar{\lambda}|_K = \lambda$ . The following lemma extends this to certain homomorphisms  $\phi : K \rightarrow M_n(K)$ .

**Lemma 4.6.** *If  $\phi : K \rightarrow M_n(K)$  represents a simple bimodule, then there exists a homomorphism  $\bar{\phi} : \bar{K} \rightarrow M_n(\bar{K})$  such that  $\bar{\phi}|_K = \phi$ .*

**Proof.** Write  ${}_1K_\phi^m \cong V(\lambda)$  for some  $\lambda^G \in A(K)$ , and write  $\lambda^G = \{\lambda_1, \dots, \lambda_m\}$ . Viewing  $\phi$  as a function from  $K$  to  $M_m(\bar{K})$ , there exists  $P \in GL_m(\bar{K})$  such that  $\phi(x) = P \operatorname{diag}(\lambda_1(x), \dots, \lambda_m(x)) P^{-1}$  for all  $x \in K$ . Lift each  $\lambda_i$  to  $\bar{\lambda}_i : \bar{K} \rightarrow \bar{K}$ , and define  $\bar{\phi}$  by the formula

$$\bar{\phi}(x) = P \operatorname{diag}(\bar{\lambda}_1(x), \dots, \bar{\lambda}_m(x)) P^{-1}. \quad (22)$$

Then one easily checks that  $\bar{\phi}$  is a lift of  $\phi$ .  $\square$

The above result obviously extends to semisimple bimodules by induction, but we will only need to apply it in the case where  $V$  is simple.

**Theorem 4.7.** *Let  $\lambda^G, \mu^G \in A(K)$ . Then  $V(\lambda) \otimes V(\mu)$  is semisimple.*

**Proof.** If  $K$  is finite, then each of  $\lambda$  and  $\mu$  is an automorphism of  $K$ , and  $V(\lambda) \otimes V(\mu) \cong V(\lambda\mu)$  is simple. So we may assume that  $K$  is infinite. Enumerate the elements of  $\lambda^G$  and  $\mu^G$  as  $\{\lambda_1, \dots, \lambda_m\}$  and  $\{\mu_1, \dots, \mu_n\}$  respectively, and let  $\bar{\lambda}_i$  and  $\bar{\mu}_j$  be lifts of  $\lambda_i$  and  $\mu_j$  to automorphisms of  $\bar{K}$ . If we write  $V(\lambda) \cong {}_1K_\phi^m$  and  $V(\mu) \cong {}_1K_\psi^n$ , then the previous lemma shows

that there are lifts  $\bar{\phi} : \bar{K} \rightarrow M_m(\bar{K})$  and  $\bar{\psi} : \bar{K} \rightarrow M_n(\bar{K})$ , such that  $\bar{\phi} \sim \text{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_m)$  and  $\bar{\psi} \sim \text{diag}(\bar{\mu}_1, \dots, \bar{\mu}_n)$ . It follows from Lemma 4.5 and an elementary calculation that  $\bar{\phi} \otimes \bar{\psi} \sim \text{diag}(\bar{\lambda}_i \bar{\mu}_j : 1 \leq i \leq m, 1 \leq j \leq n)$ .

For each pair  $(i, j)$ , let  $v_{ij} = \bar{\lambda}_i \bar{\mu}_j|_K$ . Then  $v_{ij} \in \text{Emb}(K)$  and  $v_{ij}^G \in \Lambda(K)$ . Moreover, an easy calculation shows that  $\bar{\phi} \otimes \bar{\psi}|_K = \phi \otimes \psi$ , and from this we conclude that  $\phi \otimes \psi \sim_{\bar{K}} \text{diag}(v_{ij})$ . Partition the multiset  $\{v_{ij}\}$  into a union of disjoint orbits, counting multiplicities, say  $\{v_{ij}\} = \bigcup_{k=1}^t (m_k) v_k^G$ , where  $(m_k) v_k^G$  means  $m_k$  copies of  $v_k^G$ . Let  $V = \bigoplus_{k=1}^t V(v_k)^{(m_k)}$  and write  $V \cong {}_1 K_\theta^{mn}$  for some  $\theta$ . Then by construction we have that  $\phi \otimes \psi \sim_{\bar{K}} \theta$ ; by Lemma 2.4  $\phi \otimes \psi \sim \theta$ , so that  $V(\lambda) \otimes V(\mu) \cong \bigoplus_{k=1}^t V(v_k)^{(m_k)}$  is semisimple.  $\square$

The above theorem yields a presentation for the ring  $K_0^B(K)$  by generators and relations. We distinguish between the trivial simple bimodule  $K$  which corresponds to  $\{\text{Id}_K\} \in \Lambda(K)$  and acts as the identity of  $K_0^B(K)$ , and the nontrivial simple bimodules  $\{V(\lambda) : \lambda^G \neq \{\text{Id}_K\}\}$ .

**Corollary 4.8.** Write  $\Lambda(K) = \{\text{Id}_K\} \cup (\bigcup_{i \in I} \lambda_i^G)$  as a union of disjoint orbits, and for each pair  $i, j$ , write  $V(\lambda_i) \otimes V(\lambda_j) \cong K^{(\alpha_{ij})} \oplus (\bigoplus_{l \in I} V(\lambda_l)^{(\alpha_{ijl})})$  for nonnegative integers  $\alpha_{ij}, \alpha_{ijl}$ . Then  $K_0^B(K)$  is isomorphic to the quotient of  $\mathbb{Z}\langle x_i : i \in I \rangle$  by the ideal  $I$  generated by  $\{x_i x_j - (\sum_{l \in I} \alpha_{ijl} x_l + \alpha_{ij}) : i, j \in I\}$ .

The following example illustrates how one can use Theorem 4.7 to find an explicit presentation for  $K_0^B(K)$ .

**Example 4.9.** Let  $p$  be an odd prime and let  $K = \mathbb{Q}(\rho)$ , where  $\rho$  is a real  $p$ th root of 2. As in Example 3.9,  $\text{Emb}(K)$  is partitioned into two orbits:  $\text{Emb}(K) = \{\text{Id}_K\} \cup \lambda^G$ , where  $\lambda$  is the embedding defined by  $\lambda(\rho) = \zeta \rho$ . Now,  $\text{Aut}(K(\zeta)/K)$  is cyclic of order  $p - 1$ ; let  $\sigma$  be a generator for  $\text{Aut}(K(\zeta)/K)$ . To be precise, let  $\sigma(\zeta) = \zeta^n$ , where  $n$  is a multiplicative generator for  $(\mathbb{Z}/p\mathbb{Z})^*$ . There are obvious lifts of  $\lambda$  and  $\sigma$  to automorphisms of  $\bar{K}$ ; we abuse notation and denote these lifts by  $\lambda$  and  $\sigma$  as well.

There are exactly two simple bimodules up to isomorphism: The trivial bimodule  $K$ , and the  $p - 1$ -dimensional bimodule  $V(\lambda)$ . In order to calculate the ring structure on  $K_0^B(K)$ , we must decompose  $V(\lambda) \otimes V(\lambda)$  as a direct sum of simples.

If we write  $V(\lambda) \cong {}_1 K_\phi^{p-1}$ , then  $\phi \sim_{\bar{K}} \text{diag}(\sigma^i \lambda : 0 \leq i \leq p - 2)$ . Hence  $\phi \otimes \phi \sim_{\bar{K}} \text{diag}(\sigma^i \lambda \sigma^j \lambda : 0 \leq i, j \leq p - 2)$ . So, we must count the number of times that  $\sigma^i \lambda \sigma^j \lambda|_K = \text{Id}_K$ .

We compute:

$$\sigma^i \lambda \sigma^j \lambda(\rho) = \sigma^i \lambda \sigma^j(\zeta \rho) = \sigma^i \lambda(\zeta^{n^j} \rho) = \sigma^i(\zeta^{n^{j+1}} \rho) = \zeta^{n^{i+j+1}} \rho.$$

So, we must have  $n^{i+j+1} \equiv 0 \pmod{p}$ . Since  $(n, p) = 1$ , this only happens when  $n^j + 1 \equiv 0 \pmod{p}$ , and since  $n$  is a multiplicative generator for  $(\mathbb{Z}/p\mathbb{Z})^*$ , this only happens for  $j = (p - 1)/2$ . For this value of  $j$ , we see that  $\sigma^i \lambda \sigma^{(p-1)/2} \lambda|_K = \text{Id}_K$  for all  $i$ ; in particular, there are exactly  $p - 1$  copies of the trivial bimodule as a summand of  $V(\lambda) \otimes V(\lambda)$ .

The rest is a dimension count: Since  $\dim V(\lambda) \otimes V(\lambda) = (p - 1)^2$  and  $V(\lambda) \otimes V(\lambda) \cong K^{(p-1)} \oplus V(\lambda)^{(t)}$ , it follows that  $t = p - 2$ ; i.e.  $V(\lambda) \otimes V(\lambda) \cong K^{(p-1)} \oplus V(\lambda)^{(p-2)}$ .

From this we conclude that  $K_0^B(K) \cong \mathbb{Z}[x]/(x^2 - (p - 2)x - (p - 1))$ .  $\square$

We conclude this section with a brief discussion of an alternative, “naive” approach to the Grothendieck ring of  $\mathbf{Vect}(K)$ . Namely, one could consider the free abelian group on isomorphism classes in  $\mathbf{Vect}(K)$ , modulo only those relations induced by direct sums (instead of all exact sequences). We denote this ring by  $K_0^\oplus(K)$ . Our aim is to show that, while  $K_0^B(K)$  is computable in many cases,  $K_0^\oplus(K)$  is an intractable object of study. We begin with a definition.

**Definition 4.10.** A higher  $k$ -derivation of order  $m$  (or an  $m$ -derivation) on  $K$  is a sequence of  $k$ -linear maps  $\mathbf{d} = \{d_0, d_1, \dots, d_m\}$ , such that  $d_l(xy) = \sum_{i+j=l} d_i(x)d_j(y)$  for all  $x, y \in K$ . (In particular  $d_0 : K \rightarrow K$  is an endomorphism and  $d_1$  is a  $d_0$ -derivation.) We denote the set of all  $n$ -derivations by  $HS_n(K)$ , and the set of all higher derivations (of all orders) by  $HS(K)$ . We refer the reader to [4, Section 27] for more information on higher derivations. (Note that our definition is slightly more general, in that [4] assumes that  $d_0 = \text{Id}_K$ ).

Note that  $HS(K)$  can be made into an abelian semigroup with identity as follows: Given  $\mathbf{d} = \{d_0, \dots, d_m\}$  and  $\mathbf{d}' = \{d'_0, \dots, d'_n\}$ , we define  $\mathbf{d} \cdot \mathbf{d}' = \{\delta_0, \dots, \delta_{m+n}\}$ , where  $\delta_l = \sum_{i+j=l} d_i d'_j$ . (Here we set  $d_i = 0$  for  $i > m$  and  $d'_j = 0$  for  $j > n$ .) The above operation actually makes  $HS(K)$  a group, but we will not need this fact below.

Given  $\mathbf{d} = \{d_0, d_1, \dots, d_m\}$ , we define a map  $\phi(\mathbf{d}) : K \rightarrow M_{m+1}(K)$  by

$$\phi(\mathbf{d})(x) = \begin{pmatrix} d_0(x) & d_1(x) & \dots & d_m(x) \\ 0 & d_0(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_1(x) \\ 0 & \dots & 0 & d_0(x) \end{pmatrix}. \quad (23)$$

That is,  $\phi(\mathbf{d})(x)$  is an upper triangular Toeplitz matrix, whose entry on the  $i$ th superdiagonal is  $d_i(x)$ . The fact that  $\mathbf{d} \in HS_m(K)$  is precisely the condition that  $\phi(\mathbf{d})$  is a homomorphism. It is fairly easy to see that the two-sided vector space  $V(\mathbf{d}) = {}_1K_{\phi(\mathbf{d})}^{m+1}$  is indecomposable in  $\mathbf{Vect}(K)$ . Conversely, if  $\phi : K \rightarrow M_{m+1}(K)$  is a homomorphism such that  $\phi(x)$  is an upper triangular Toeplitz matrix for all  $x \in K$ , then  $\mathbf{d} = \{d_0, \dots, d_m\} \in HS_m(K)$ , where  $d_i(x)$  is the  $i$ th superdiagonal of  $\phi(x)$ .

It follows readily that there is an abelian semigroup homomorphism  $\Psi : \mathbb{Z}[HS(K)] \rightarrow K_0^\oplus(K)$ . However,  $\Psi$  is in general neither injective nor surjective, and is also not a ring homomorphism.

For instance, let  $\mathbf{d} = \{d_0, d_1\}$  and let  $\mathbf{d}' = \{d_0, xd_1\}$  for some  $x \in K^*$ . Then conjugating  $\phi(\mathbf{d})$  by  $\text{diag}(x, 1)$  shows that  $V(\mathbf{d}) \cong V(\mathbf{d}')$  and so  $\Psi$  is not injective. Similarly, let  $V = {}_1K_{\phi}^3$ , where

$$\phi(x) = \begin{pmatrix} d_0(x) & d_1(x) & d_1(x) \\ 0 & d_0(x) & 0 \\ 0 & 0 & d_0(x) \end{pmatrix}.$$

Then  $V$  is indecomposable but is not in the image of  $\Psi$ , so  $\Psi$  is not surjective.

The fact that  $\Psi$  is not a ring homomorphism is easy: If  $\mathbf{d} \in HS_m(K)$  and  $\mathbf{d}' \in HS_n(K)$ , then  $\mathbf{d} \cdot \mathbf{d}' \in HS_{m+n}(K)$  and so the left dimension of  $V(\mathbf{d} \cdot \mathbf{d}')$  is  $m + n + 1$ . On the other hand, the left dimension of  $V(\mathbf{d}) \otimes V(\mathbf{d}')$  is  $(m + 1)(n + 1)$ .

The above remarks show that in general  $K_0^\oplus(K)$  is a more intractable object of study than  $K_0^B(K)$ , and that its structure depends on significantly subtler arithmetic properties of the field  $K$ .

## 5. Representatives for equivalence classes of matrix homomorphisms

Let  $\phi : K \rightarrow M_n(K)$  be a homomorphism. In this final section, we consider the problem of finding a representative for the  $\sim$ -equivalence class of  $\phi$  that has a particularly “nice” form.

For example, suppose that  $\phi(y)$  has all of its eigenvalues in  $K$  for some  $y \in K$ . Then there exists  $P \in GL_n(K)$  such that  $P\phi(y)P^{-1}$  is in Jordan canonical form. Let  $\lambda_1, \dots, \lambda_t$  be the distinct eigenvalues of  $\phi(y)$ , with corresponding multiplicities  $m_1, \dots, m_t$ . For each  $i$ , let  $n_{i,1}, \dots, n_{i,s_i}$  be the sizes of the  $\lambda_i$ -Jordan blocks of  $P\phi(y)P^{-1}$ . Then [2, Section VIII.2] implies the following result. (We say that an  $m \times n$  matrix  $A$  is *generalized upper triangular Toeplitz* if it is of the form  $\begin{pmatrix} 0 & T \end{pmatrix}$  or  $\begin{pmatrix} T \\ 0 \end{pmatrix}$ , where  $T$  is an upper triangular Toeplitz matrix.)

**Theorem 5.1.** *Assume the above notation. For all  $x \in K$ ,*

$$P\phi(x)P^{-1} = \text{diag}(\phi_1(x), \dots, \phi_t(x)), \quad (24)$$

where each  $\phi_i(x)$  is an  $m_i \times m_i$ -block matrix of the form  $\phi_i(x) = (T_{ipq}(x))$ , where  $T_{ipq}(x)$  is a generalized upper triangular Toeplitz matrix of size  $n_{i,p} \times n_{i,q}$ .

The above theorem uses nothing more than the description of the set of all matrices which commute with a given matrix in Jordan canonical form; in particular it does *not* use the additional information that  $\phi$  is a homomorphism, or that the matrices in  $\text{im } \phi$  also commute with each other. Consequently one can often find a better representation than the one afforded by Theorem 5.1.

**Example 5.2.** Suppose that  $\phi : K \rightarrow M_3(K)$  is such that  $\phi(y) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$  for some  $y \in K$ . Then  $\phi(y)$  is in Jordan canonical form, so Theorem 5.1 shows that there exist functions  $a, b, c, d, e : K \rightarrow K$  such that

$$\phi(x) = \begin{pmatrix} a(x) & b(x) & c(x) \\ 0 & a(x) & 0 \\ 0 & d(x) & e(x) \end{pmatrix}. \quad (25)$$

Writing out the condition that  $\phi$  is a homomorphism shows that each of  $a$  and  $e$  are (nonzero) homomorphisms from  $K$  to  $K$ . If we conjugate  $\phi(x)$  by the matrix  $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ , we see that

$\phi(x) \sim \psi(x) = \begin{pmatrix} a(x) & c(x) & b(x) \\ 0 & e(x) & d(x) \\ 0 & 0 & a(x) \end{pmatrix}$ . If we let  $V = {}_1K_{\psi}^3$ , then the composition factors of  $V$  are  $\{{}_1K_a, {}_1K_e, {}_1K_a\}$ .

Suppose first that  $a = e$ . Then the fact that  $\psi$  is a homomorphism implies that  $b(x_1x_2) = a(x_1)b(x_2) + c(x_1)d(x_2) + b(x_1)a(x_2)$  for all  $x_1, x_2 \in K$ . Since  $b(x_1x_2) = b(x_2x_1)$  we can equate terms and get that  $c(x_1)d(x_2) = c(x_2)d(x_1)$ . If  $c \neq 0$ , then choosing  $x_2$  so that  $c(x_2) \neq 0$ , we see that  $d(x) = \alpha c(x)$ , where  $\alpha = d(x_2)/c(x_2)$ . If  $\alpha \neq 0$ , then we can conjugate  $\psi$  by  $Q = \text{diag}(1, 1, \alpha)$  to conclude that  $\phi \sim \begin{pmatrix} a & c & \frac{1}{\alpha}b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix}$ . If  $\alpha = 0$ , then  $d = 0$  and so  $\phi \sim \begin{pmatrix} a & c & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$ . Finally, if  $a \neq e$  then the fact that there are no nontrivial extensions between nonisomorphic



simples shows that  $\phi \sim \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & e \end{pmatrix}$ . Thus we conclude that  $\phi$  is equivalent to a homomorphism as in Theorem 5.1 that is also upper triangular.

Motivated by the above example, we may ask whether a homomorphism  $\phi$  is always equivalent to an upper triangular homomorphism or, ideally, an upper triangular homomorphism of the form (24). Assuming that the matrices in  $\text{im } \phi$  have their eigenvalues in  $K$ , the answer to the first question is “yes” [3, p. 100]. We shall prove that, under certain additional assumptions, the matrices in  $\text{im } \phi$  have upper triangular Toeplitz diagonals. We then derive a sufficient condition for an affirmative answer to the second question. We begin with some elementary reductions.

Given  $V \in \text{Vect}(K)$ , let  $S_1, \dots, S_t$  be a complete list of the pairwise nonisomorphic composition factors of  $V$ . Since  $\text{Ext}^1(S_i, S_j) = 0$  for  $i \neq j$ , we see that  $V \cong V_1 \oplus \dots \oplus V_t$ , where each  $V_i$  has each of its composition factors isomorphic to  $S_i$ . Now, if  $\phi$  represents  $V$  and  $\phi_i$  represents  $V_i$  for each  $i$ , then it is clear that  $\phi \sim \text{diag}(\phi_1, \dots, \phi_t)$ . Thus it suffices to consider the case where the composition factors of  ${}_1K_\phi^n$  are pairwise isomorphic. We shall further assume that the simple composition factor of  ${}_1K_\phi^n$  is isomorphic to  ${}_1K_a$  for some  $a : K \rightarrow K$ ; we shall say that  $\phi$  is *a-homogeneous* in this case.

**Lemma 5.3.** *If  $\phi : K \rightarrow M_n(K)$  is a-homogenous for some  $a : K \rightarrow K$ , then  $\phi$  is equivalent to an upper triangular homomorphism with each diagonal entry equal to  $a$ .*

**Proof.** We proceed by induction on  $n$ , the case  $n = 1$  being trivial. Let  $V = {}_1K_\phi^n$ . Then  ${}_1K_a$  is a sub-bimodule of  $V$ , generated as a left subspace by a single vector  $v$ . Choose a basis for  $V$  containing  $v$  and order it so that  $v$  occurs last; then we see that, in this basis,  $V \cong {}_1K_\psi^n$ , where  $\tilde{\phi} \sim \begin{pmatrix} \psi & \theta \\ 0 & a \end{pmatrix}$  for some  $\psi : K \rightarrow M_{n-1}(K)$ . Now,  ${}_1K_\psi^{n-1}$  is also *a-homogeneous* and so by induction is equivalent to an upper triangular homomorphism with each diagonal entry equal to  $a$ . The result follows.  $\square$

**Theorem 5.4.** *Let  $\phi : K \rightarrow M_n(K)$  be a-homogeneous for some  $a : K \rightarrow K$ . Then there exist higher derivations  $\mathbf{d}_1, \dots, \mathbf{d}_t$ , each of whose 0th components is equal to  $a$ , such that*

$$\phi \sim \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1t} \\ 0 & A_{22} & \dots & A_{2t} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{tt} \end{pmatrix}, \quad (26)$$

where  $A_{ii}(x) = \phi(\mathbf{d}_i)(x)$  and  $A_{ij}(xy) = \sum_{l=1}^t A_{il}(x)A_{lj}(y)$  for all  $x, y \in K$ .

**Proof.** The fact that  $A_{ij}(xy) = \sum_l A_{il}(x)A_{lj}(y)$  follows because  $\phi$  is a homomorphism; the key is to show that the diagonal matrices  $A_{ii}$  have the stated form. By the previous lemma, we may assume without loss of generality that  $\phi$  is upper triangular. Write  $\phi = (\phi_{ij})$ , where  $\phi_{ii} = a$  for all  $i$  and  $\phi_{ij} = 0$  for  $i > j$ . Let  $i_1 \leq \dots \leq i_q$  be all of the indices for which  $\phi_{i_k, i_k+1} = 0$ . Then we can partition  $\phi$  into blocks of size  $i_1, i_2 - i_1, \dots, i_q - i_{q-1}, n - i_q$ . If we let  $\phi_l$  denote the  $l$ th diagonal block in this partition, then each  $\phi_l$  has the properties that each of its diagonal entries is equal to  $a$ , and none of its first superdiagonal entries is identically 0.

Replacing  $\phi$  by  $\phi_i$  we may assume without loss of generality that  $\phi_{i,i+1}$  is not identically 0 for any  $i$ . After these reductions, we see that the theorem is trivially true when  $n = 1$  or 2, so we assume without loss of generality that  $n \geq 3$ . If we expand out  $\phi_{i,i+2}(xy)$  using the fact that  $\phi$  is a homomorphism and  $\phi_{ij} = 0$  for  $i > j$ , we obtain

$$\phi_{i,i+2}(xy) = \phi_{ii}(x)\phi_{i,i+2}(y) + \phi_{i,i+1}(x)\phi_{i+1,i+2}(y) + \phi_{i,i+2}(x)\phi_{i+2,i+2}(y) \quad (27)$$

and a similar equation for  $\phi_{i,i+2}(yx)$ . Substituting  $\phi_{ii} = \phi_{i+2,i+2}$  and using the fact that  $\phi(xy) = \phi(yx)$ , we can simplify the resulting equations to obtain

$$\phi_{i,i+1}(x)\phi_{i+1,i+2}(y) = \phi_{i,i+1}(y)\phi_{i+1,i+2}(x) \quad (28)$$

for all  $x, y \in K$ . If we choose  $y$  such that  $\phi_{i,i+1}(y) \neq 0$ , then we have that  $\phi_{i+1,i+2}(x) = \alpha_i \phi_{i,i+1}(x)$  for all  $x \in K$ , where  $\alpha_i = \phi_{i+1,i+2}(y)/\phi_{i,i+1}(y)$ . Note also that  $\alpha_i \neq 0$  for any  $i$  since we know that  $\phi_{i+1,i+2}$  is not identically 0.

Let  $b = \phi_{12}$  and let  $\beta_i = \prod_{j \leq i} \alpha_j$ , so that

$$\phi = \begin{pmatrix} a & b & & & * \\ 0 & a & \beta_1 b & & \\ \vdots & 0 & a & \ddots & \\ \vdots & & \ddots & \ddots & \beta_{n-2} b \\ 0 & \dots & \dots & 0 & a \end{pmatrix}.$$

Choose  $y \in K$  with  $b(y) \neq 0$ . An elementary calculation shows that  $(\phi(y) - a(y)I_n)^{n-1}$  is the matrix whose only nonzero entry is  $\beta_1 \cdots \beta_{n-2} b(y)^{n-1}$  in its  $(1, n)$ -position. This shows that the minimal polynomial for  $\phi(y)$  is  $(X - a(y))^n$ , so that the Jordan canonical form for  $\phi(y)$  is a single block of size  $n$ . If  $P \in GL_n(K)$  is such that  $P\phi(y)P^{-1}$  is in Jordan canonical form, then Theorem 5.1 shows that  $P\phi(x)P^{-1}$  is an upper triangular Toeplitz matrix with diagonal equal to  $a(x)$  for all  $x \in K$ . Thus there exists a higher derivation  $\mathbf{d}$  such that  $P\phi P^{-1} = \phi(\mathbf{d})$ . This shows that  $\phi$  is equivalent to a matrix of the form (26).  $\square$

One may ask under what circumstances it is possible to obtain the best of both worlds: That is, when can we conclude that  $\phi$  is equivalent to an upper triangular representation as in (26), and also have each  $A_{ij}$  be a generalized upper triangular Toeplitz matrix as in Theorem 5.1? Since the Toeplitz condition arises out of commuting with a matrix in Jordan canonical form, the following would be a sufficient condition:

- (\*) Given a homomorphism  $\phi$ , there exists  $y \in K$  and  $P \in GL_n(K)$  such that  $P\phi(y)P^{-1}$  is in Jordan canonical form and  $P\phi(x)P^{-1}$  is upper triangular for all  $x \in K$ .

If  $\phi$  is an upper triangular homomorphism, then of course condition (\*) is satisfied if there exists  $y \in K$  and an upper triangular  $P \in GL_n(K)$  such that  $P\phi(y)P^{-1}$  is in Jordan canonical form.

Condition (\*) is not automatic for a given  $y$  and  $\phi$ . The following example illustrates that, given  $y$ , there may be no  $P$  such that  $P\phi(y)P^{-1}$  is in Jordan canonical form and  $P\phi(x)P^{-1}$  is upper triangular.

**Example 5.5.** Let  $\mathbf{d} = \{d_0, d_1, d_2\}$  be a 2-derivation, and assume that  $d_1 \neq 0$  and that there exists a  $y \in K$  such that  $d_1(y) = 0$ ,  $d_2(y) \neq 0$ . Define  $\phi : K \rightarrow M_3(K)$  by

$$\phi(x) = \begin{pmatrix} d_0(x) & d_2(x) & d_1(x) \\ 0 & d_0(x) & 0 \\ 0 & d_1(x) & d_0(x) \end{pmatrix}.$$

We claim there does not exist a basis in which  $\phi(y)$  has Jordan canonical form and the image of  $\phi$  is upper triangular. To establish this claim, we describe every  $P \in GL_3(K)$  in which the image of  $P\phi P^{-1}$  is upper triangular, and show that  $P\phi(y)P^{-1}$  is not in Jordan canonical form for any such  $P$ .

Since  $d_1 \neq 0$ , it is not hard to see that the only simultaneous eigenvectors for  $\text{im } \phi$  are in  $W = \text{span}(0, 1, 0)$ . Similarly, the only simultaneous eigenvectors for  $\text{im } \phi$  acting on  $K^3/W$  are in  $\text{span}\{(0, 0, 1) + W\}$ . From this we conclude that, if  $\mathcal{B}$  is a basis with  $\text{im } P\phi P^{-1}$  upper triangular, then

$$\mathcal{B} = \{(0, f_1, 0), (0, f_2, f_3), (f_4, f_5, f_6) : f_1, f_3, f_4 \neq 0\}.$$

For such a basis  $\mathcal{B}$ , we have

$$P\phi(x)P^{-1} = \begin{pmatrix} d_0(x) & \frac{f_4}{f_3}d_1(x) & \left(\frac{f_6f_3-f_4f_2}{f_1f_3}\right)d_1(x) + \frac{f_4}{f_1}d_2(x) \\ 0 & d_0(x) & \frac{f_3}{f_1}d_1(x) \\ 0 & 0 & d_0(x) \end{pmatrix}.$$

By construction,  $P\phi(y)P^{-1}$  is not in Jordan canonical form.

Note that higher derivations satisfying the given hypotheses do exist. For example, let  $K$  be the quotient field of  $k[x, y, z]/(xy - z^2)$ , where  $k$  is a field of characteristic 2. In [7, Example 1.2 and Theorem 1.5], a nontrivial  $\mathbf{d} \in HS_2(K)$  is constructed such that  $d_1(x - z) = 0$  and  $d_2(x - z) = x$ .

**Definition 5.6.** Let  $A$  be an  $n \times n$  upper triangular matrix with single eigenvalue  $\lambda$ , and let the Jordan canonical form of  $A$  have block sizes  $n_1 \geq n_2 \geq \cdots \geq n_p$ . For each  $i \leq n$ , let  $A_i$  be the matrix consisting of the first  $i$  rows and columns of  $A$ . We say that  $A$  is *Jordan-ordered* if, for all  $i \leq n$ , the dimension of the eigenspace of  $A_i$  is  $j$ , where  $j$  is the smallest integer such that  $n_1 + \cdots + n_j \geq i$ .

**Example 5.7.** Let  $A = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ , so that the Jordan canonical form of  $A$  has blocks of size 2

and 1. Then  $A$  is not Jordan-ordered, because the dimension of the eigenspace of  $A_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$  is 2 and not 1.

It is not hard to see that, if  $A$  is in Jordan canonical form, then  $A$  is Jordan-ordered if and only if the Jordan blocks of  $A$  are arranged in decreasing size.

The following is our main result concerning Jordan-ordered matrices.

**Theorem 5.8.** If  $A \in M_n(K)$  is Jordan-ordered, then there exists an upper triangular  $P \in GL_n(K)$  such that  $PAP^{-1}$  is Jordan-ordered and is in Jordan canonical form.

We begin with a preliminary lemma.

**Lemma 5.9.** Suppose that  $A \in M_{n+1}(K)$  has a single eigenvalue  $\lambda$  of multiplicity  $n$ . If

$$A = \begin{pmatrix} & & a_1 \\ & B & \vdots \\ 0 & \dots & 0 & \lambda \end{pmatrix} \quad (29)$$

with  $B \in M_n(K)$  in Jordan canonical form, then there exists an upper triangular  $P \in GL_n(K)$  such that  $PAP^{-1}$  is in Jordan canonical form.

**Proof.** Since  $A$  has the single eigenvalue  $\lambda$ , the Jordan canonical form for  $A$  must be

$$\begin{pmatrix} & & 0 \\ & B & \vdots \\ & & 0 & a \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}, \quad (30)$$

with  $a = 0$  or  $1$ . We give the proof when  $a = 1$ , the case  $a = 0$  being similar and left to the reader.

Let  $E_A$  denote the eigenspace of  $A$  and suppose that  $\dim E_A = m + 1$ . Since  $e_{n+1} \in E_A$ , we can take a basis for  $E_A$  containing it; moreover elementary calculations then allow us to assume that the final entry of all other basis elements is  $0$ . Thus  $E_A$  has a basis of the form

$$\{(0, \dots, 0, 1), (c_{11}, \dots, c_{1n}, 0), \dots, (c_{m1}, \dots, c_{mn}, 0)\}.$$

Since the last  $m$  of these vectors are eigenvectors for  $A$ , we see that  $(a_1, \dots, a_n)$  must be a solution to the system of equations

$$\begin{aligned} c_{11}x_1 + \dots + c_{1n}x_n &= 0 \\ \vdots \\ c_{m1}x_1 + \dots + c_{mn}x_n &= 0 \end{aligned} \quad (31)$$

and that

$$\{(c_{11}, \dots, c_{1n}), \dots, (c_{m1}, \dots, c_{mn})\}$$

is a set of  $m$  linearly independent eigenvectors of  $B$ . Because  $a = 1$  we see that the dimension of the eigenspace  $E_B$  of  $B$  is also  $m + 1$ , and we note that  $e_n \in E_B$ . Since  $B$  is in Jordan canonical form, the matrix

$$\begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mn} \end{pmatrix} \quad (32)$$

has  $n - m - 1$  of its columns equal to  $0$ , and its final column cannot be equal to  $0$  since  $e_n$  is an eigenvector for  $B$ . Thus (31) can be viewed as a system of  $m$  equations in  $m + 1$  variables, say  $x_{i_1}, \dots, x_{i_{m+1}} = x_n$ . Since the rows of (32) are linearly independent, some subset of  $m$  columns of (32) is linearly independent. Thus the solution space of (31) is 1-dimensional. On the other hand, since  $A$  has the given Jordan canonical form,  $(x_{i_1}, \dots, x_{i_{m+1}}) = (0, 0, \dots, 1)$  must be a solution to (31). Thus we conclude that  $(a_{i_1}, \dots, a_{i_{m+1}}) = (0, 0, \dots, c)$  for some  $c \in K$ .

Consider the system

$$(\lambda I_n - B) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ 0 \end{pmatrix}.$$

Since  $B$  is in Jordan canonical form, the image of left multiplication by  $\lambda I_n - B$  has each of its  $i_1, \dots, i_{m+1}$ -components equal to 0, and also has dimension  $n - m - 1$ . Since  $a_{i_1} = \dots = a_{i_m} = 0$ , we see that there is a solution  $y_1 = b_1, \dots, y_n = b_n$ . Let  $\vec{b}$  be the column vector  $(b_1, \dots, b_n)^T$ ; then an elementary calculation shows that, if  $P = \begin{pmatrix} I_n & \vec{b} \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(K)$ , then

$$PAP^{-1} = \begin{pmatrix} & & 0 \\ & B & \vdots \\ & & 0 \\ 0 & \dots & c \\ & & \lambda \end{pmatrix}.$$

It follows, since the Jordan canonical form for  $A$  is (30), that  $c \neq 0$ . Conjugating by  $\text{diag}(1, \dots, 1, 1/c)$  finishes the proof.  $\square$

**Proof of Theorem 5.8.** We proceed by induction on  $n$ , the case  $n = 1$  being trivial. Since  $A$  is upper triangular,  $e_n$  is an eigenvector for  $A$ . If  $A_{n-1}$  denotes the matrix obtained by deleting the last row and column from  $A$ , then by induction there exists an upper triangular  $Q \in GL_{n-1}(K)$  such that  $QA_{n-1}Q^{-1}$  is Jordan-ordered and in Jordan canonical form. Let  $R = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \in GL_n(K)$ ; conjugating  $A$  by  $R$  then gives

$$RAR^{-1} = \begin{pmatrix} & & a_1 \\ & B & \vdots \\ & & a_n \\ 0 & \dots & 0 & \lambda \end{pmatrix},$$

where  $B$  is the Jordan ordered, Jordan canonical form for  $A_{n-1}$ .

Let the Jordan canonical form for  $A$  have blocks of sizes  $n_1 \geq \dots \geq n_p$ . If  $n_p = 1$ , then  $B$  has blocks of sizes  $n_1, \dots, n_{p-1}$ , and the Jordan canonical form for  $A$  is  $\begin{pmatrix} B & 0 \\ 0 & \lambda \end{pmatrix}$ . By Lemma 5.9, there is an upper triangular  $T \in GL_n(K)$  with  $TRAR^{-1}T^{-1}$  Jordan-ordered and in Jordan canonical form. Thus the theorem follows with  $P = TR$  in this case.

If  $n_p > 1$ , then  $B$  has blocks of sizes  $n_1, \dots, n_{p-1}, p-1$  and the block of size  $n_p - 1$  occurs at the bottom of  $B$ . Thus the Jordan canonical form for  $A$  is

$$\begin{pmatrix} & & 0 \\ & B & \vdots \\ & & 0 \\ 0 & \dots & 1 \\ & & \lambda \end{pmatrix},$$

and again letting  $T$  be as in Lemma 5.9, we see that  $PAP^{-1}$  is Jordan-ordered and in Jordan canonical form for  $P = TR$ .  $\square$

Combining Theorems 5.4 and 5.8, we can state a sufficient condition for a homomorphism  $\phi : K \rightarrow M_n(K)$  to be equivalent to an upper triangular homomorphism which is generalized upper triangular Toeplitz. We state the result in the case where  $\phi$  is  $a$ -homogenous for some  $a : K \rightarrow K$ .

**Corollary 5.10.** *Let  $\phi$  be  $a$ -homogeneous, and let  $\psi \sim \phi$ , where  $\psi$  is a homomorphism in the form (26). If  $\psi(y)$  is Jordan-ordered for some  $y \in K$ , then*

$$\phi \sim \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1s} \\ 0 & T_{22} & \dots & T_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{ss} \end{pmatrix}, \quad (33)$$

where each  $T_{ij}(x)$  is generalized upper triangular Toeplitz.

In particular there exist higher derivations  $\mathbf{d}_1, \dots, \mathbf{d}_s$  such that  $T_{ii} = \phi(\mathbf{d}_i)$ , although these derivations may be different than those in (26).

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